Appl. Math. Lett. Vol. 6, No. 1, pp. 31–34, 1993
0893-9659/93 \$6.00 + 0.00

A GLOBAL STABILIZATION OF A CLASS OF NONLINEAR SYSTEMS BY MEANS OF AN OBSERVER

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(Received May 1992)

Abstract—This paper deals with the stabilization problem. More precisely, we give a class of feedback linearisable systems which are stabilizable by a feedback law using the state estimate given by an observer.

1. INTRODUCTION

We consider systems which are observable for any input of the form:

$$
\begin{aligned}
\dot{x}_1 &= x_2\\
&\;\vdots\\
\dot{x}_{k_1-1} &= x_{k1}\\
\dot{x}_{k_1} &= \varphi_1(x) + u_1\psi_1(x)\\
\dot{x}_{k_1+1} &= x_{k_1+2}\\
&\;\vdots\\
\dot{x}_{k_1+k_2-1} &= x_{k_1+k_2}\\
\dot{x}_{k_1+k_2} &= \varphi_2(x) + u_2\psi_2(x)\\
&\;\vdots\\
x_{k_1+\cdots+k_{p-1}+1} &= x_{k_1+\cdots+k_{p-1}+2}\\
&\;\vdots\\
\dot{x}_{k_1+\cdots+k_p-1} &= x_{k_1+\cdots+k_p}\\
\dot{x}_{k_1+\cdots+k_p} &= \varphi_p(x) + u_p\psi_p(x)\\
y &= \begin{bmatrix} x_1\\ x_{k_1+1}\\ \vdots\\ x_{k_1+\cdots+k_{p-1}+1}\end{bmatrix}
\end{aligned}
\tag{Σ}
$$

with $\sum_{i=1}^{p} k_i = n$, and all the φ_i's and the ψ_i's are global Lipschitz functions. In a condensed form we can write:

$$
\begin{aligned}
\dot{x} &= Ax + \Phi(x) + \Psi(x)u,\\
y &= Cx,
\end{aligned}
\tag{Σ}
$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX where

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_p \end{pmatrix} \text{ with } A_i = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \text{ and } \Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_p \end{pmatrix} \text{ with } \Phi_i = \begin{pmatrix} 0 \\ \vdots \\ \varphi_i \end{pmatrix};$$

$$\Psi = (\Psi_1, \dots, \Psi_p) \text{ with } \Psi_i = \begin{pmatrix} 0 \\ \vdots \\ \psi_i \\ \vdots \\ 0 \end{pmatrix}; \; u = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix};$$

$$C = \begin{pmatrix} C_1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & C_p \end{pmatrix} \text{ with } C_i = (1\ 0 \dots 0).$$

An observer for these systems does exist and is given in [1] for the single output case and in [2] for the multi-output case. It possesses the following expression:

$$\dot{\hat{x}} = A\hat{x} + \Phi(\hat{x}) + \Psi(\hat{x})u - S_\theta^{-1}C^\top(C\hat{x} - y) \tag{O}$$

where S_θ is a symmetric positive definite matrix, solution of:

$$\theta S_\theta + A^\top S_\theta + S_\theta A = C^\top C \tag{1}$$

for θ sufficiently large and $\hat{x}$ is the estimate of the state x.

NOTE. ($^\top$) denotes the transposition of matrices.

The coefficients of $(S_\theta)_{ij}$ are given by:

$$(S_\theta)_{ij} = \frac{1}{\theta^{i+j-1}}\alpha_{ij} \tag{2}$$

and (α_{ij}) is a symmetric positive definite matrix which does not depend on θ [1,2].

It should be noted that (Σ) is already in the normal form stated in [2]. It is also pointed out that if u is bounded, then (O) is an exponential observer for (Σ).

That is, if we set $\varepsilon = \hat{x}(t) - x(t)$, the error equation:

$$\dot{\varepsilon} = (A - S_\theta^{-1}C^\top C)\varepsilon + (\Phi(\hat{x}) - \Phi(x)) + (\Psi(\hat{x}) - \Psi(x))u \tag{E}$$

is stable and converges exponentially to zero. More precisely, $\forall u \in L^\infty(\mathbb{R}^+)$, $\exists k > 0$ s.t. $\|\varepsilon\| \le ke^{-\theta t/3}$. Suppose that $\psi_i(x) \neq 0$; $\forall x \in \mathbb{R}^n$; $i = 1, \dots, p$.

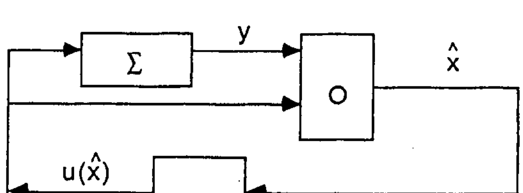
Then, according to the form of (Σ), it is clear that one can choose coefficients a_{ij} such that:

$$u(x) = \begin{pmatrix} u_1(x) \\ \vdots \\ u_p(x) \end{pmatrix} \text{ where } u_i(x) = \frac{\sum_{j=1}^{k_i} a_{ij}x_j - \varphi_i(x)}{\psi_i(x)} \tag{3}$$

becomes a linearising and stabilizing feedback law of (Σ).

2. SEPARATION PRINCIPLE

We are now going to interest ourselves in the separation principle which consists in the study of the stability of the following closed-loop system:



In other words, we consider the problem of stability of the couple (observer, controlled system) when the feedback is taken as a function of the estimate of the state given by the observer. To resolve this problem, let us consider the following augmented system:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \Phi(\hat{x}) + \Psi(\hat{x})u(\hat{x}) - S_\theta^{-1}C^\top(C\hat{x} - y), \\ \dot{\varepsilon} &= (A - S_\theta^{-1}C^\top C)\varepsilon + (\Phi(\hat{x}) - \Phi(\hat{x} - \varepsilon)) + (\Psi(\hat{x}) - \Psi(\hat{x} - \varepsilon))u(\hat{x}), \end{aligned} \tag{S}$$

which becomes:

$$\begin{aligned} \dot{\hat{x}} &= \tilde{A}\hat{x} - S_\theta^{-1}C^\top C\varepsilon, \\ \dot{\varepsilon} &= (A - S_\theta^{-1}C^\top C)\varepsilon + (\Phi(\hat{x}) - \Phi(\hat{x} - \varepsilon)) + (\Psi(\hat{x}) - \Psi(\hat{x} - \varepsilon))u(\hat{x}), \end{aligned} \tag{S}$$

where

$$\tilde{A} = \begin{pmatrix} \tilde{A}_1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \tilde{A}_p \end{pmatrix}; \qquad \tilde{A}_i = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ a_{i1} & \dots & \dots & a_{ik_i} \end{pmatrix}.$$

The coefficients a_{ij}'s are those which appear in the formula (3) of $u(x)$.

THEOREM 1. *Let $u(x)$ be the feedback given in (3). Assume that $\sup_{x\in\mathbb{R}^n} \|u(x)\| < \infty$. Then (S) is globally asymptotically stable.*

PROOF OF THEOREM. Consider $V(\hat{x},\varepsilon) = V_1(\varepsilon) + V_2(\hat{x})$, where $V_1(\varepsilon) = \varepsilon^\top S_\theta \varepsilon$ and $V_2(\hat{x}) = \hat{x}^\top P\hat{x}$, P being a symmetric positive definite matrix such that $\tilde{A}^\top P + P\tilde{A} = -I$.

Let us show that V is a Lyapunov function for the system (S):

(i)

$$\begin{aligned} \frac{d}{dt}(V_1(\varepsilon)) &= \frac{d}{dt}(\varepsilon^\top S_\theta\varepsilon) = 2\varepsilon^\top S_\theta\dot{\varepsilon} \\ &= 2\varepsilon^\top S_\theta A\varepsilon - 2(C\varepsilon)^2 + 2\varepsilon^\top S_\theta(\Phi(\hat{x}) - \Phi(\hat{x}-\varepsilon)) + 2\varepsilon^\top S_\theta(\Psi(\hat{x}) - \Psi(\hat{x}-\varepsilon))u(\hat{x}) \\ &= -\theta\varepsilon^\top S_\theta\varepsilon - (C\varepsilon)^2 + 2\varepsilon^\top S_\theta(\Phi(\hat{x}) - \Phi(\hat{x}-\varepsilon)) + 2\varepsilon^\top S_\theta(\Psi(\hat{x}) - \Psi(\hat{x}-\varepsilon))u(\hat{x}) \\ &\quad \text{(by (1))}. \end{aligned}$$

Denote by $\|x\|_{S_\theta}$ the norm $(x^\top S_\theta x)^{1/2}$ and using the Schwarz inequality, we obtain:

$$\frac{d}{dt}\left((\|\varepsilon\|_{S_\theta})^2\right) \le -\theta\,(\|\varepsilon\|_{S_\theta})^2 + 2\|\varepsilon\|_{S_\theta}\,\|\Phi(\hat{x}) - \Phi(\hat{x}-\varepsilon)\|_{S_\theta} + 2\|\varepsilon\|_{S_\theta}\,\|\Psi(\hat{x}) - \Psi(\hat{x}-\varepsilon)\|_{S_\theta} r_0$$
$$\text{(where } r_0 = \sup_{x\in\mathbb{R}^n} \|u(x)\|\text{)}.$$

Now, using the particular form of S_θ, Φ, Ψ and the fact that Φ, Ψ are global Lipschitz, we obtain: $\|\Phi(\hat{x}) - \Phi(\hat{x}-\varepsilon)\|_{S_\theta} \le \lambda_1\|\varepsilon\|_{S_\theta}$ and $\|\Psi(\hat{x}) - \Psi(\hat{x}-\varepsilon)\|_{S_\theta} \le \lambda_2\|\varepsilon\|_{S_\theta}$ for some constants λ_1, λ_2 which does not depend on θ, for $\theta \ge 1$. Then,

$$\frac{d}{dt}\|\varepsilon\|_{S_\theta} \le -\frac{\theta}{2}\|\varepsilon\|_{S_\theta} + \lambda_1\|\varepsilon\|_{S_\theta} + \lambda_2\|\varepsilon\|_{S_\theta} r_0.$$

Choose θ such that $\theta/2 - \lambda_1 - \lambda_2 r_0 = \gamma_\theta > 0$. We get:

$$\frac{d}{dt}\|\varepsilon\|_{S_\theta} \le -\gamma_\theta\|\varepsilon\|_{S_\theta}.$$

So:

$$\|\varepsilon\|_{S_\theta} \le e^{-\gamma_\theta t}\|\varepsilon(0)\|_{S_\theta}.$$

(ii)

$$\begin{aligned} \frac{d}{dt}(V_2(\hat{x})) &= \frac{d}{dt}(\hat{x}^\top P\hat{x}) = 2\hat{x}^\top P\dot{\hat{x}} \\ &= \hat{x}^\top P\tilde{A}\hat{x} + \hat{x}^\top\tilde{A}^\top P\hat{x} - 2\hat{x}^\top PS_\theta^{-1}C^\top C\varepsilon = -\hat{x}^\top\hat{x} - 2\hat{x}^\top PS_\theta^{-1}C^\top C\varepsilon \\ &\le -\alpha\hat{x}^\top P\hat{x} - 2\hat{x}^\top PS_\theta^{-1}C^\top C\varepsilon = -\alpha V_2 - 2\hat{x}^\top PS_\theta^{-1}C^\top C\varepsilon \\ &\quad \text{(for some constants } \alpha > 0\text{)}. \end{aligned}$$

Hence, using the Schwarz inequality we have:

$$\frac{d}{dt}(V_2) \le -\alpha V_2 + 2\rho_\theta \|\varepsilon\|_{S_\theta} V_2^{1/2},$$
$$2\frac{d}{dt}(V_2^{1/2}) \le -\alpha V_2^{1/2} + 2\rho_\theta \|\varepsilon\|_{S_\theta}.$$

Choose θ such that $\gamma_\theta > \alpha/2$.

This ends the proof.

We have proved the separation principle for the above class of systems (Σ) in the case where the u_i's are bounded and where the φ_i's and ψ_i's are global Lipschitz functions. In the case where the boundedness of u is not required, we have:

COROLLARY 2. *Assume that the ψ_i's are non-zero constants and φ_i's are global Lipschitz. Then (S) is globally asymptotically stable.*

Note that for the proof of Theorem 1, the boundedness of $u(x)$ has been used in the term $(\Psi(\hat{x}) - \Psi(\hat{x} - \varepsilon))u(\hat{x})$. Here, this term vanishes out. Indeed, (S) takes the following form:

$$\begin{aligned} \dot{\hat{x}} &= \tilde{A}\hat{x} - S_\theta^{-1}C^\top C\varepsilon, \\ \dot{\varepsilon} &= (A - S_\theta^{-1}C^\top C)\varepsilon + \Phi(\hat{x}) - \Phi(\hat{x} - \varepsilon). \end{aligned} \tag{S}$$

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